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ASYMPTOTIC EXPANSION OF THE GREEN FUNCTION OF AN INTERNAL-WAVE EQUATION FOR $t \to \infty$

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The Green function of an internal-wave equation for a horizontally uniform, stratified fluid layer $Z_- < z < Z_+$, where Z_- and/or Z_+ can become infinity, is considered. For the case of waveguide propagation of internal waves [i.e., for $N^2(z) = gd\ln\rho_0(z)/dz \to 0$ at $|z| \to \infty$] and the additional condition that the function $N(z)$ has one local maximum, the asymptotic behavior of the Green function for $t \to \infty$ is constructed. The well-known representation of the Green function as a sum of normal waves is used for this. These waves are replaced by their WKB-asymptotics [1], after which Poisson's summation formula is used, and the resultant integrals are calculated by the steady-state method.

1. Formulation of the Problem. In region Ω ($-\infty < x, y < \infty$, and $Z_- < z < Z_+$, where Z_-, Z_+ can turn to $\mp\infty$, respectively) the Green function $G(t, \sqrt{x^2+y^2}, z, z_0)$ of the internal-wave equation is considered:

$$\frac{\partial^2}{\partial t^2}(G_{xx}+G_{yy}+G_{zz}) + N^2(z)(G_{xx}+G_{yy}) = \delta(t)\delta(x,y)\delta(z-z_0). \tag{1.1}$$

The uniqueness of solution is ensured by the condition $G = 0$ at $t < 0$ and by the requirement that G decreases as $x^2+y^2+z^2 \to \infty$. The Dirichlet boundary condition or the Newman condition is set on the corresponding boundary for finite Z_+ or Z_-. The asymptotic representation of G for $t \to \infty$ is to be found. The problem considered is the original one for a wide range of issues related to the behavior at large times of the fields of internal waves excited by distributed oscillating sources.

The asymptotic representation of G for Eq. (1.1) in the half-space $z > 0$ for $N^2(z) = B^2 z$ ($B = \text{const}$) was found in [2, 3] proceeding from approximate [2] and exact [3] expressions for the Green function. Based on these results, Borovikov [4] put forward the hypothesis that, in the general case, for $t \to \infty$ the asymptotic expansion of Green function has the form

$$G(t,r,z,z_0) \approx \sum_p \frac{A_p}{\sqrt{t\omega_p}} \sin(t\omega_p + \psi_p) + o(t^{-1/2}), \tag{1.2}$$

where A_p and ω_p are functions of r, z, and z_0; and ψ_p are constants. Expression (1.2) is analogous to the radial (geometrical-optical) description of the wave field for the Helmholtz equation $\Delta u + k^2 n^2(x,y)u = 0$:

$$u(t,x,y) \approx \sum_p A_p \exp iks_p,$$

with the difference that the large parameter determining the characteristic period of oscillations for the asymptotic expansion of the Green function is the time t rather than the wavenumber k. This means, in particular, that the internal-wave field excited by a δ-shaped source becomes more and more short-wave when x, y, and z are fixed and t increases.

In [4], an eikonal equation for $\omega_p = \omega_p(r,z,z_0)$ and a transport equation for $A_p = A_p(r,z,z_0)$ were derived and their solution that provides the desired asymptotic representation of G is presented.

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Asymptotic representation (1.2) is justified and generalized in the present paper for the waveguide propagation of internal waves, i.e., under the assumption that either $Z_\pm$ are finite or (in the case of infinite Z_- and(or) Z_+) $N^2(z) \to 0$ as $z \to \infty$. The above assumptions satisfied, the vertical spectral problem corresponding to Eq. (1.1) has a discrete spectrum. The function G is constructed using the method of separation of variables (see, for example, [5]) and is presented as the sum (3.1) of normal waves.

Let us also assume that $N^2(z)$ has one local maximum. Then, the WKB-asymptotics can be written for normal waves. The resultant series is transformed into a sum of integrals using Poisson's summation formula. Calculation of these integrals by the steady-state method yields the desired asymptotic representation of the Green function.

A similar approach was developed in [6] to pass from a mode description of the fields in acoustic waveguides to a radial description. However, there is a significant difference in situations for acoustic and internal gravitational waves. For acoustic waves, the radial method (and its modifications in the case of fields with caustics) gives asymptotic representations of the field in each fixed vicinity of the source for sufficiently large k. Therefore, a relationship between the two methods for field description is established in [6], each of them having an independent justification. In the case of internal waves, asymptotic series (1.2) is valid only for sufficiently large t. Within the framework of the approach developed in [4] we can say nothing about the Green function for limited t. Thus, constructing asymptotic series (1.2) proceeding from the Green function represented as a sum of normal waves is a single available method for justification of this asymptotic expansion at the moment.

Let us consider the physical sense of the above constraints. The condition of waveguide propagation is fulfilled in all cases of internal-wave propagation in water (both under natural conditions and in laboratory experiments). In addition, rejection of this condition will apparently change neither the qualitative behavior of the Green function nor the algorithm for constructing its asymptotic representation described below.

The condition of uniqueness of the local maximum of the Brunt–Väisälä frequency is satisfied in laboratory experiments in which the water-density distribution in a tank is close to that of a two-layer medium, and under natural conditions of wave propagation on a shelf or in the deep ocean when the seasonal thermocline is absent or the influence of the depth pycnocline can be ignored. At the same time, this condition is essential for our asymptotic representation. If it is not satisfied, the rays can be captured by the second local maximum. The trajectory of the ray emanating from the source O is a discontinuous function of the parameter ω (see below), and the algorithm below for determining the asymptotics needs a considerable generalization.

2. Algorithm for Determining the Asymptotics. Let us formulate an algorithm for determining the functions ω_p and A_p in asymptotic series (1.2). For this, we introduce the notion of a ray. Each ray emanating from the source O ($r = 0$ and $z = z_0$) is characterized by the value ω [$\omega \leqslant N(z_0)$] and is defined as a curve that satisfies the equation

$$\frac{dz}{dr} = \delta \frac{\omega}{\sqrt{N^2(z) - \omega^2}}, \tag{2.1}$$

where $\delta = \pm 1$. This equation is integrated for fixed δ from point O until point $P = (r, z)$ falls on the boundary of region Ω or on the horizon z on which $N(z) = \omega$; after this we change the sign of δ and continue the integration.

Let us call the point $P = (r, z)$, at which the integration direction is altered, i.e., δ is changed, the turning point. Obviously, at the turning point we have $z = z_\pm$, where

$$z_+ = \min \begin{cases} Z_+, \\ z_+(\omega), \end{cases} \qquad z_- = \max \begin{cases} Z_-, \\ z_-(\omega). \end{cases} \tag{2.2}$$

Here $z_\pm(\omega)$ [$z_-(\omega) < z_+(\omega)$] are roots of the equation $N(z) = \omega$.

If the turning point P lies on the boundary $z = Z_\pm$ of region Ω, a ray that goes out of this point makes the same angle with the boundary as the incoming ray. Let us call such a turning point the point of ray reflection. If $N(z) = \omega$ at P, the ray direction tends to vertical when approaching this point, as is seen from (2.1). We call such a turning point the point of ray return.

If we again get to the turning point during the next integration, we change once again the sign of δ, etc.

To formalize this procedure, we assume that

$$I(z_1, z_2) = \int_{z_1}^{z_2} \frac{\sqrt{|N^2(z) - \omega^2|}}{\omega}\, dz; \tag{2.3}$$

$$I = I(z_-, z_+). \tag{2.4a}$$

We also introduce the integrals I_p which depend on the integer parameter p:
for $p = 0$

$$I_0 = |I(z_0, z)|; \tag{2.4b}$$

for $p > 0$

$$I_p = \begin{cases} I(z_0, z_+) + (p-1)I + I(z_-, z) & \text{for even } p, \\ I(z_0, z_+) + (p-1)I + I(z, z_+) & \text{for odd } p; \end{cases} \tag{2.4c}$$

for $p < 0$

$$I_p = \begin{cases} I(z_-, z_0) + (|p|-1)I + I(z, z_+) & \text{for even } p, \\ I(z_-, z_0) + (|p|-1)I + I(z_-, z) & \text{for odd } p. \end{cases} \tag{2.4d}$$

The p-fold reflected rays $L_p(\omega)$ and $L_{-p}(\omega)$ are defined by the formula

$$r = I_p(z, \omega). \tag{2.5}$$

For given r, z_0, and z, the function $\omega_p = \omega_p(r, z, z_0)$ in (1.2) is defined as a solution of Eq. (2.5). Since I_p is a monotone decreasing function of ω, this solution is unique if ever exists. For every $r > 0$, there is only a finite number of p values for which the function ω_p is defined, i.e., a finite number of terms in (1.2), this number growing with increasing r. An exception is only the case $z = z_0 = \hat{z}$ where $\hat{z}$ is the value of z for which $N(z)$ reaches a maximum. In this case, Eq. (2.5) has solutions for every p, and sum (1.2) has an infinite number of terms.

According to [4], the function $A_p(r, z, z_0)$ in (1.2) has the form

$$A_p(r, z, z_0) = A(r, z, z_0, \omega_p), \tag{2.6a}$$

where

$$A(r, z, z_0, \omega(r, z, z_0)) = \frac{(2\pi)^{-3/2}}{[(N^2(z) - \omega^2)(N^2(z_0) - \omega^2)]^{1/4}} \left(\frac{-1}{r\omega} \frac{\partial \omega}{\partial r} \right)^{1/2}. \tag{2.6b}$$

It remains to determine the phase shift ψ_p. Let $\psi_0 = \pi/4$ and let the turning point $P = (r, z_\pm)$ correspond to the phase increment:

$$\delta_\pm = \begin{cases} -\pi, & \text{if} \quad z_\pm = Z_\pm, \quad \text{i.e., } P \text{ is the point of ray reflection,} \\ -\pi/2, & \text{if} \quad z_\pm = z_\pm(\omega), \quad \text{i.e., } P \text{ is the point of ray return.} \end{cases} \tag{2.7}$$

The phase shift ψ_p is the sum of the initial phase shift ψ_0 and phase increments at all turning points of the ray L_p:

$$\psi_p = \begin{cases} \dfrac{\pi}{4} + \dfrac{|p|}{2}(\delta_+ + \delta_-) & \text{for even } p, \\ \dfrac{\pi}{4} + \delta_+ + \dfrac{p-1}{2}(\delta_+ + \delta_-) & \text{for odd } p > 0, \\ \dfrac{\pi}{4} + \delta_- + \dfrac{|p|-1}{2}(\delta_- + \delta_-) & \text{for odd } p < 0. \end{cases} \tag{2.8}$$

Asymptotic expansion (1.2) is not uniform. In particular, it is inapplicable in the following cases:

(a) In the vicinity of the source horizon $z = z_0$, i.e., at the limit $\omega_0(r, z, z_0) \to 0$, when the term in (1.2) with $p = 0$ tends to infinity;

(b) In the vicinity of the points of ray return, i.e., at the limit $N^2(z) - \omega_p^2 \to 0$, when two neighboring terms in sum (1.2) corresponding to two rays of which one approaches the return point and the other leaves it tend to infinity [a similar situation occurs as $N^2(z_0) - \omega_p^2 \to 0$];

(c) In the vicinity of the rays L_p for which the return point tends to the boundary of region Ω, i.e., for which $z_-(\omega_p) \to Z_-$ or $z_+(\omega_p) \to Z_+$, and, hence, the phase increments δ_- or δ_+ are discontinuous, according to (2.7).

It will be proved below that the uniform asymptotic representation is constructed as follows:

In case (a) [4], in the term with $p = 0$ tending to infinity in sum (1.2) the function $\sin(t\omega_0 + \pi/4)/\sqrt{t\omega_0}$ is replaced by $\sqrt{\pi/2}J_0(t\omega_0)$ (J_0 is a Bessel function).

In case (b) [4], the two terms tending to infinity

$$\frac{A(r, z, z_0, \omega_p)}{\sqrt{t\omega_p}} \sin(t\omega_p + \psi_p) + \frac{A(r, z, z_0, \omega_{p+1})}{\sqrt{t\omega_{p+1}}} \sin(t\omega_{p+1} + \psi_p - \pi/2)$$

are replaced by a combination of the Airy function and its derivative:

$$\sqrt{\pi}t^{-1/3} \sin(t\xi + \psi_p - \pi/4)\,\zeta^{1/4}(B_2 + B_1)\,\mathrm{Ai}(-t^{2/3}\zeta) + \sqrt{\pi}t^{-2/3} \cos(t\xi + \psi_p - \pi/4)\,\zeta^{-1/4}(B_2 - B_1)\,\mathrm{Ai}'(-t^{2/3}\zeta), \tag{2.9a}$$

where

$$\xi = \frac{\omega_p + \omega_{p+1}}{2}, \quad \zeta = \frac{3}{4}(\omega_p - \omega_{p+1})^{2/3}, \quad B_1 = \frac{A(r, z, z_0, \omega_p)}{\sqrt{t\omega_p}}, \quad B_2 = \frac{A(r, z, z_0, \omega_{p+1})}{\sqrt{t\omega_{p+1}}}. \tag{2.9b}$$

In case (c), the phase increments $\delta_\pm$, which are discontinuous for values $\omega = \omega_\pm$ such that $z_\pm(\omega) = Z_\pm$, are replaced by the smoothed functions $\hat{\delta}_\pm(tq_p, \omega)$. To define the latter, we introduce the functions $Q_\pm(k, \omega, z)$:

$$Q_+(k, \omega, z) = \mathrm{sign}(z_+(\omega) - z)\left[\frac{3k}{2}\left|I(z_+(\omega), z)\right|\right]^{2/3},$$
$$Q_-(k, \omega, z) = \mathrm{sign}(z - z_-(\omega))\left[\frac{3k}{2}\left|I(z, z_-(\omega))\right|\right]^{2/3}. \tag{2.10}$$

Here, $I(z_1, z_2)$ is integral (2.3). It is easy to verify that if $N(z)$ is an analytical function in the vicinity of values of $z_\pm(\omega)$ with a non-zero derivative, then $Q_\pm(z)$ are analytical in the vicinity of $z_+(\omega)$ and $z_-(\omega)$, respectively.

We set [7, formula 10.4.69]

$$\mathrm{Ai}(-x) = M(x)\cos\theta(x), \quad \mathrm{Bi}(-x) = M(x)\sin\theta(x), \tag{2.11a}$$

where $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are Airy functions, and introduce the functions

$$\tilde{\theta}(x) = \begin{cases} 2\theta(x) & \text{for} \quad x < 0, \\ 2\theta(x) + \frac{4}{3}x^{3/2} & \text{for} \quad x > 0. \end{cases} \tag{2.11b}$$

According to [7, formula 10.4.79],

$$\lim_{|x|\to\infty} \tilde{\theta}(x) = \begin{cases} \pi & \text{for} \quad x \to -\infty, \\ \pi/2 & \text{for} \quad x \to \infty. \end{cases}$$

Let

$$\tilde{\delta}_\pm(k, \omega) = \tilde{\theta}(Q_\pm(k, \omega, Z_\pm)) - 3\pi/2. \tag{2.12a}$$

Then the smoothed functions $\hat{\delta}_\pm(t, \omega)$ have the form

$$\hat{\delta}_\pm(t, \omega) = \tilde{\delta}_\pm(tq_p, \omega). \tag{2.12b}$$

Here $q_p = |\partial I_p/\partial\omega|^{-1}$ and I_p is an integral (2.4). If the turning point $z_\pm$ is a return point, i.e., it coincides with $z_\pm(\omega)$, we have $Q_\pm(tq_p, \omega, Z_\pm) \to -\infty$ at $t \to \infty$ and $\tilde{\delta}_\pm \to -\pi/2$. If $z_\pm$ is the return point, we have $Q_\pm(tq_p, \omega, Z_\pm) \to \infty$ and $\tilde{\delta}_\pm \to -\pi$. Thus, definition (2.12) is in agreement with (2.7).

Asymptotics series (1.2) is also inapplicable when z and z_0 are on the horizon $\hat{z}$ on which $N(z)$ reaches a maximum. Then series (1.2) includes an infinite number of terms, their amplitude is bounded from below for $p \to \infty$, and the uniform asymptotic series of the Green function has a more complicated form. This case is not considered here.

3. Expansion of the Green Function in Terms of Normal Waves. We now justify the formulated algorithm. According to [5], the Green function $G(t, r, z, z_0)$ has the form

$$G(t, r, z, z_0) = \sum_{n=0}^{\infty} S_n, \tag{3.1a}$$

where

$$S_n = -\frac{1}{2\pi}\int_0^\infty J_0(kr)\sin(t\omega_n)\frac{\omega_n}{k}\varphi_n(z,k)\varphi_n(z_0,k)\,dk; \tag{3.1b}$$

and $\omega_n = \omega_n(k)$ and $\varphi_n(z,k)$ are the eigenvalues and eigenfunctions of the spectral problem:

$$\varphi'' + \frac{k^2}{\omega^2}(N^2 - \omega^2)\varphi = 0; \tag{3.2a}$$

$$\varphi(Z_\pm, k) = 0. \tag{3.2b}$$

Here, k is a free parameter and ω is a spectral parameter. The functions φ_n are normalized in the interval $Z_- < z < Z_+$ with weight $N^2(z)$:

$$\int_{Z_-}^{Z_+} N^2(z)\varphi_n^2(z,k)\,dz = 1.$$

In the limit $k \to \infty$ and for fixed n, the functions $\varphi_n(z,k)$ are concentrated in the vicinity of the value $z = \hat{z}$ in which $N(z)$ reaches a maximum and tends to zero for fixed $z \neq z_0$ as $\exp(-\text{const}\cdot k)$. The function ω_n in the limit $k \to \infty$ has the asymptotic representation

$$\omega_n \approx N(\hat{z}) - \frac{c_n}{k} + O(k^{-2}), \tag{3.3}$$

where the coefficient c_n grows with increasing n.

Our objective is to find an asymptotic representation of the function G for $t \to \infty$. For each fixed n and $t \to \infty$, the term S_n in sum (3.1a) tends to zero not more slowly than t^{-1}. Indeed, as $t \to \infty$, the steady-state point of the phase function in (3.1b) determined [taking into account oscillations of the function $J_0(kr)$] from the equation $-r + t\partial\omega_n/\partial k = 0$, with allowance for (3.3), tends to infinity as $\sqrt{t/r}$ in the limit $t \to \infty$. For sufficiently high t, it appears in the region in which the product $\varphi_n(z,k)\varphi_n(z_0,k)$ is exponentially small and is estimated as $\exp(-\text{const}\cdot k)$. Therefore, the asymptotics S_n is determined by the boundary $k = 0$ of the domain of integration in (3.1b). Since $\omega_n \sim k$ for $k \to 0$, the contribution of this point to the asymptotics S_n has the order of t^{-1}. In calculation of the asymptotic behavior of the function G in the limit $t \to \infty$, this quantity can be ignored, i.e., the summation in (3.1a) can be performed beginning with some fixed value of n. Then, since $k/\omega_n \to \infty$ uniformly with respect to k as $n \to \infty$, the WKB-asymptotics below can be used in this calculation for the eigenvalues ω_n and eigenfunctions φ_n.

4. WKB-Asymptotics of Eigenfunctions and Eigenvalues. To find the asymptotic representation of the eigenvalues, we write asymptotic representations of the solutions $u = u_\pm$ of Eq. (3.2a), which vanish for $z = Z_\pm$, respectively. Equating these two asymptotics, we obtain an equation for the WKB-asymptotics of the eigenvalues ω_n.

For $k/\omega \gg 1$, the asymptotic representation of the general solution of (3.2a) which is applicable in an interval that contains the point $z = z_-(\omega)$ and does not contain $z_+(\omega)$ has the form (see, for instance,

[1]) $u \approx (\partial Q_-(k,\omega,z)/\partial z)^{-1/2}[A_-\mathrm{Ai}(-Q_-(k,\omega,z)) + B_-\mathrm{Bi}(-Q_-(k,\omega,z))]$. In a similar manner, within an interval that contains the point $z_+(\omega)$ and does not contain $z_-(\omega)$, the general solution of Eq. (3.2a) has the asymptotic representation $u \approx (\partial Q_+(k,\omega,z)/\partial z)^{-1/2}[A_+\mathrm{Ai}(-Q_+(k,\omega,z))+ B_+\mathrm{Bi}(-Q_+(k,\omega,z))]$. Here, $Q_\pm$ are defined by formula (2.10), and $A_\pm$ and $B_\pm$ are arbitrary constants. If the function u satisfies a zero boundary condition for $z = Z_-$, we have

$$u = u_- \approx \frac{2\sqrt{\pi}A}{\sqrt{Q'_-(k,\omega,z)}}[-\sin\theta(Q_-(k,\omega,Z_-))\mathrm{Ai}(-Q_-(k,\omega,z)) + \cos\theta(Q_-(k,\omega,Z_-))\mathrm{Bi}(-Q_-(k,\omega,z))], \tag{4.1a}$$

where A is an arbitrary constant and the function $\theta(x)$ is defined by formula (2.11a). Replacing the Airy functions $\mathrm{Ai}(-Q_-(k,\omega,z))$ and $\mathrm{Bi}(-Q_-(k,\omega,z))$ by their asymptotic representations for $|Q_-| \gg 1$, we obtain a nonuniform asymptotic representation of u_- which is applicable for $z < z_+(\omega)$ and $z \neq z_-(\omega)$:

$$u = u_- \approx \begin{cases} 0 & \text{for} \quad z < z_-(\omega), \\ \dfrac{A\sin[kI(z_-,z) + \tilde{\theta}(Q_-(k,\omega,Z_-))/2 - \pi/4]}{(N^2(z) - \omega^2)^{1/4}} & \text{for} \quad z > z_-(\omega). \end{cases} \tag{4.2a}$$

Here, z_- is determined using formula (2.2), and the functions I and $\tilde{\theta}$ are found using (2.3) and (2.11).

Analogously, the solution $u = u_+$ vanishing for $z = Z_+$ has an asymptotic representation applicable in the vicinity of the point $z = z_+(\omega)$:

$$u = u_+ \approx \frac{2\sqrt{\pi}A}{\sqrt{Q'_+(k,\omega,z)}}[-\sin\theta(Q_+(k,\omega,Z_+))\mathrm{Ai}(-Q_-(k,\omega,z)) + \cos\theta(Q_+(k,\omega,Z_+))\mathrm{Bi}(-Q_+(k,\omega,z))] \tag{4.1b}$$

and the nonuniform asymptotics feasible for $z > z_-(\omega)$, $z \neq z_+(\omega)$

$$u = u_+ \approx \begin{cases} \dfrac{A\sin[kI(z,z_+) + \tilde{\theta}(Q_+(k,\omega,Z_+))/2 - \pi/4]}{(N^2(z) - \omega^2)^{1/4}} & \text{for} \quad z < z_+(\omega), \\ 0 & \text{for} \quad z > z_+(\omega). \end{cases} \tag{4.2b}$$

If ω is an eigenvalue of the spectral problem (3.2), the functions u_- and u_+ must coincide with one-place accuracy. Hence, taking into account that $I(z,z_+) = I - I(z_-,z)$ where the integral I is defined by (2.4a), we obtain an equation for the asymptotics of the eigenvalues $\omega = \omega_n$:

$$\Psi(\omega) = \frac{1}{2}[\tilde{\theta}(Q_-(k,\omega,Z_-)) + \tilde{\theta}(Q_+(k,\omega,Z_+))] + kI$$

$$= \theta(Q_-(k,\omega,Z_-)) + \theta(Q_+(k,\omega,Z_+)) + \frac{k}{\omega}\int\limits_{z_-(\omega)}^{z_+(\omega)} \sqrt{N^2(z) - \omega^2}dz = \left(n + \frac{1}{2}\right)\pi. \tag{4.3}$$

Since Q_+ and Q_- are monotone increasing functions of ω, and $\theta(x)$ is a decreasing function [7, formula 10.4.71], the function $\Psi(\omega)$ is a monotone decreasing function of ω, which tends to infinity as $\omega \to 0$ together with the integral I. Thus, the eigenvalues ω_n are uniquely defined by Eq. (4.3).

Formulas (4.1) and (4.2) are valid for the eigenfunctions $\varphi_n(k,z) = \varphi(k,z,\omega_n)$. In these formulas the constant determined from the normalization condition is

$$A = \frac{\sqrt{2}}{\omega}\left|\frac{\partial I}{\partial\omega}\right|^{-1/2}, \tag{4.2c}$$

where I is integral (2.4a) and $\omega = \omega_n$.

Let $\varphi(k,z,\omega_n)\varphi(k,z_0,\omega_n) = U(k,z,z_0,\omega_n)$. Then, it follows from (4.2) that the function $U(k,z,z_0,\omega)$ has the following nonuniform asymptotic representation:

for $z_-(\omega) < \min(z, z_0)$ and $\max(z, z_0) < z_+(\omega)$

$$U \approx \frac{2[e^{ikI(z_0,z)} + e^{-ikI(z_0,z)} + e^{ikI_{-1}+i\tilde{\delta}_-(k,\omega)} + e^{ikI_1+i\tilde{\delta}_+(k,\omega)}]}{[(N^2(z)-\omega^2)(N^2(z_0)-\omega^2)]^{1/4}\omega^2\left|\dfrac{\partial I}{\partial \omega}\right|}; \tag{4.4}$$

and for $\min(z, z_0) < z_-(\omega)$ or $z_+(\omega) < \max(z, z_0)$

$$u = 0.$$

Here, the integrals $I(z_0, z)$ and $I_\pm$ and the functions $\tilde{\delta}_\pm(k, \omega)$ are defined by formulas (2.4) and (2.12a).

This asymptotics is not applicable for values of ω close to $N(z)$ or to $N(z_0)$, i.e., for z or z_0 close to $z_\pm(\omega)$.

In this case the asymptotics expressed in terms of Airy functions should be used for the functions $\varphi(k, z, \omega)$ and $\varphi(k, z_0, \omega)$, and thus, for U. For instance, asymptotic representation (4.1a) is used for the function $\varphi(k, z, \omega)$ at z close to $z_-(\omega)$

5. The Green Function Asymptotics. If the function $f(\xi)$ is even, the Poisson's summation formula can be written as

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \sum_{m=-\infty}^{\infty} \int_0^{\infty} f(\xi) \exp(2\pi i m \xi)\, d\xi.$$

Replacing the terms S_n in sum (3.1a) by their WKB-asymptotics, we obtain with accuracy up to $O(t^{-1})$

$$G(t, r, z, z_0) = \sum_{m=-\infty}^{\infty} T_m, \tag{5.1a}$$

where

$$T_m = -\frac{1}{2\pi} \int_0^{\infty} d\xi \int_0^{\infty} J_0(kr) \exp(2\pi i m\xi) \sin t\omega U(k, z, z_0, \omega) \frac{\omega}{k}\, dk. \tag{5.1b}$$

Here $\omega = \omega(\xi)$ is found from the equation $\xi = (1/2\pi)\Psi(\omega) = (1/2\pi)(\tilde{\theta}_- + \tilde{\theta}_+ + 2kI) - 1/2$ $[\tilde{\theta}_\pm = \tilde{\theta}(Q_\pm(k, \omega, Z_\pm))]$. In the limit $\xi \to 0$ we have $\omega(\xi) \to \max N(z) = N(\hat{z}) = \widehat{N}$. Transforming to the integration variables $q = k/t$ and in (5.1a) ω, we obtain

$$T_m = -\frac{(-1)^m t}{2\pi^2} \int_0^{\widehat{N}} d\omega \int_0^{\infty} \Phi(t, q, \omega, z, z_0)\Omega(t, q, \omega, z, z_0) dq, \tag{5.2}$$

where

$$\Phi(t, q, \omega, z, z_0) = \frac{\sin t\omega}{\omega} J_0(tqr) \exp(2imtqI) U(tq, z, z_0, \omega);$$

$$\Omega(t, q, \omega, z, z_0) = \omega^2 \exp(im(\tilde{\theta}_- + \tilde{\theta}_+)) \left|\partial I/\partial\omega\right| (1 + H(tq, \omega));$$

$$H(tq, \omega) = \frac{1}{2}\left(\frac{\partial\tilde{\theta}_+}{\partial\omega} + \frac{\partial\tilde{\theta}_-}{\partial\omega}\right) \Big/ \left(tq\frac{\partial I}{\partial\omega}\right) \quad \left(\tilde{\theta}_\pm = \tilde{\theta}(Q_\pm(tq, \omega, Z_\pm))\right).$$

In the limit $t \to \infty$ the function H tends to zero as $t^{-1/3}$. Therefore, we set $H = 0$ in the subsequent asymptotic representation of the Green function. The function Φ is a rapidly oscillating function of t, and its derivatives with respect to q and ω are on the order of t as $t \to \infty$. The function Ω can be considered a slowly changing function of q and ω, since its derivatives are on the order of $t^{2/3}$.

When calculating the asymptotics T_m, we assume first that for the given r, z, and z_0 the values of ω at stationary steady-state points of the function Φ are different from $N(z)$ and $N(z_0)$. Then, the nonuniform asymptotic representation (4.4) can be used for U, i.e., it is assumed that

$$T_m = -\frac{t}{4\pi^2}\int_0^\infty dq \int_0^{N_-} \frac{J_0(tqr)e^{im[\widetilde{\delta}_-(tq,\omega)+\widetilde{\delta}_+(tq,\omega)+2tqI]}}{\left((N^2(z)-\omega^2)(N^2(z_0)-\omega^2)\right)^{1/4}}\frac{\sin t\omega}{\omega}$$

$$\times\left(e^{itqI(z_0,z)} + e^{-itqI(z_0,z)} + e^{itqI_{-1}+i\widetilde{\delta}_-(\omega)} + e^{itqI_1+i\widetilde{\delta}_+(\omega)}\right)d\omega,$$

where $N_- = \min(N(z_0), N(z))$.

If this expression is substituted into (5.1b), the resultant sum is

$$G = \sum_{p=-\infty}^{\infty} L_p.$$

Here

$$L_p = -\frac{t}{4\pi^2}\int_0^\infty dq \int_0^{N_-} \frac{J_0(tqr)\cos\left(tqI_p(z,z_0,\omega)+\psi_p(tq,\omega)-\pi/4\right)}{\left((N^2(z)-\omega^2)(N^2(z_0)-\omega^2)\right)^{1/4}}\frac{\sin t\omega}{\omega}\,d\omega;$$

I_p are integrals (2.4), the functions ψ_p are defined by formula (2.8) in which the constants $\delta_\pm$ are replaced by the smoothed functions $\widetilde{\delta}_\pm(tq,\omega)$.

Let us find asymptotic representations of the integrals L_p in the limit $t \to \infty$. We begin with calculation of L_0. Since in this case $\psi_0 = \pi/4$, the integration with respect to q is reduced to a tabular integral [8, formula 6.672.6], and

$$L_0 = -\frac{1}{2\pi^2}\int_{\omega_0}^{N_-} \frac{\sin t\omega d\omega}{\left((N^2(z)-\omega^2)(N^2(z_0)-\omega^2)\right)^{1/4}\omega\sqrt{r^2-I_0^2}},$$

where ω_0 is a root of Eq. (2.5) for $p = 0$. Calculating the asymptotics of L_0 in the limit $t \to \infty$, we obtain expression (2.6) for the function $A_0(r,z,z_0)$.

The above asymptotics becomes invalid as $z \to z_0$, i.e., $\omega_0 \to 0$. To find the asymptotics of L_0 in this case, we write this function in the form

$$L_0 = -\frac{1}{2\pi^2}\int_{\omega_0}^{N_-} \frac{\sin t\omega}{\sqrt{\omega^2-\omega_0^2}}\,W(\omega)\,d\omega.$$

Here, the function

$$W(\omega) = \frac{1}{\omega\left((N^2(z)-\omega^2)(N^2(z_0)-\omega^2)\right)^{1/4}}\sqrt{\frac{\omega^2-\omega_0^2}{r^2-I_0^2(\omega)}}$$

is regular uniformly with respect to $z - z_0$ for small ω. In the limit $t \to \infty$ the main term of the asymptotics of L_0 is given by

$$L_0 = -\frac{1}{2\pi^2}\,W(\omega_0)\int_{\omega_0}^{\infty} \frac{\sin t\omega}{\sqrt{\omega^2-\omega_0^2}}\,d\omega = -\frac{1}{4\pi}\,J_0(t\omega_0)W(\omega_0).$$

Calculating $W(\omega_0)$, we obtain the algorithm described in Section 2 for finding asymptotic representation of the Green function as $z \to z_0$.

We now consider the asymptotics of L_p for $p \neq 0$. Replacing $J_0(tqr)$ by the asymptotics of this function for $tqr \gg 1$, we have the product of three trigonometric functions under the integration sign. We represent them as the sum of exponents and use the steady-state method. The stationary points of the phase function are determined from the equations

$$\frac{\partial}{\partial\omega}(\omega - qr + qI_p) = \frac{\partial}{\partial q}(\omega - qr + qI_p) = 0, \quad \text{i.e.} \quad r = I_p = I_p(z, z_0, \omega), \quad 1 + q\partial I_p/\partial\omega = 0.$$

The first of these equations coincides with (2.5) and defines the function $\omega_p(r, z, z_0)$, and the second is used to find the function $q_p = q_p(z, z_0, \omega_p) = |\partial I_p/\partial\omega|^{-1}$. Obviously, the phase function value at the steady point is ω_p.

Calculating L_p by the steady-state method, we obtain expression (1.2) in which ω_p is determined from Eq. (2.5), A_p from (2.6), and ψ_p from (2.8), where the smoothed functions $\delta_\pm$ determined from (2.12) are replaced by the constants $\widehat{\delta}_\pm(tq_p, \omega_p)$.

The steady-state method is inapplicable if some value of ω_p is close to $N(z)$ or $N(z_0)$. In this case, the nonuniform asymptotics of the function U must not be used for the function U in the corresponding integral T_m [see (5.2)]. The asymptotics of U expressed in terms of Airy functions (see Section 4) should be used. As a result, we find an expression for T_m in the form of a threefold integral whose phase function has two close stationary points. The uniform asymptotics of such integrals is considered, for instance, in [9]. Using these results, we obtain formulas (2.9), which are applicable for ω_p that are close to $N(z)$ or to $N(z_0)$.

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